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COMBINED FORCED AND FREE CONVECTION FLOW ON VERTICAL SURFACES

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NOMENCLATURE

- f, dimensionless stream function;
- Gr_x, Grashof number, $g\beta\Delta Tx^3/\nu^2$;
- Gr_x^*, modified Grashof number, $g\beta qx^4/k\nu^2$;
- g, acceleration due to gravity;
- h, local heat transfer coefficient, $q/\Delta T$,
- k, thermal conductivity;
- Nu_x, local Nusselt number, hx/k;
- Pr, Prandtl number, ν/κ;
- q, heat flux constant;
- Re_x, Reynolds number, Ux/ν;
- T, temperature;
- T_0, ambient temperature;
- T_w, local plate temperature;
- U, free stream velocity;
- u, streamwise velocity;
- v, transverse velocity;
- x, streamwise coordinate;
- y, transverse coordinate;
- ΔT, $T_w - T_0$;
- β, coefficient of thermal expansion;
- θ, dimensionless temperature;
- η, pseudo-similarity variable;
- κ, coefficient of thermal conductivity;
- ν, kinematic viscosity;
- ξ, dimensionless streamwise coordinate;
- ψ, stream function.

1. INTRODUCTION

THIS note is concerned with the flow envisaged when a uniform stream U flows along a semi-infinite plate extending vertically upwards with its leading edge horizontal. Also present are favourable buoyancy forces introduced by a uniform surface heat flux q from the plate relative to the surrounding ambient temperature T_0. In many types of exchangers this condition is closer to actual conditions than the isothermal condition which has been discussed by Lloyd and Sparrow [1]. Consequently the work that follows presents results to supplement those of the above named authors. In particular these authors acknowledged the criticisms made by Merkin [2] with respect to attempted series solutions about the basic free and forced convection flows as outlined by Szewczyk [3]. Essentially the error in [3] could be attributed to the choice of a parameter expansion as opposed to the correct choice, namely a coordinate expansion. In this light it is seen that account must be taken of difficulties, which arise in the form of eigen-solutions, in the perturbed free convection solution. These difficulties are associated with the asymptotic nature of this solution at a distance far removed from the leading edge and indeed reflect the solutions' unawareness of the precise location of the leading edge (see Stewartson [4]). In [2] a dimensional analysis of the governing equations yields the choice of non-dimensional coordinate as

$$\frac{Gr_x}{Re_x^2} = \frac{g\beta(T_w - T_0)x}{U^2}.$$

This coordinate reflects the basic concepts of the problem namely a transition from a perturbed forced convection flow near the leading edge to a perturbed free convection flow downstream.

In the constant flux problem the non-dimensional parameters governing the flow comprise the local Reynolds number $Re_x = Ux/\nu$, Grashof number $Gr_x = g\beta\Delta Tx^3/\nu^2$ and the Nusselt number $Nu_x = q_x/k\Delta T$ where $\Delta T = T_w - T_0$ and T_w is the local temperature of the plate. A dimensional analysis of the basic equations yields the non-dimensional coordinate for this situation as

$$\xi = \left(\frac{Gr_x^2 Nu_x^2}{Re_x^5}\right)^{\frac{1}{3}} = \left(\frac{Gr_x^{*2}}{Re_x^5}\right)^{\frac{1}{3}} = \left(\frac{g^2\beta^2q^2\nu}{k^2U^5}\right)^{\frac{1}{3}} . x$$

where Gr_x^* is known as the modified Grashof number. ξ clearly reflects the transition from a perturbed forced convection flow at small ξ to a perturbed free convection flow at large ξ. It can be demonstrated that solutions for the cases of small ξ and large ξ involve expansions in $\xi^{\frac{3}{2}}$ near the leading edge and $\xi^{-\frac{3}{2}}$ for downstream. However, as pointed out in [1], the concept of local similarity is inapplicable at large ξ and attention must be focussed on information appropriate to small ξ.

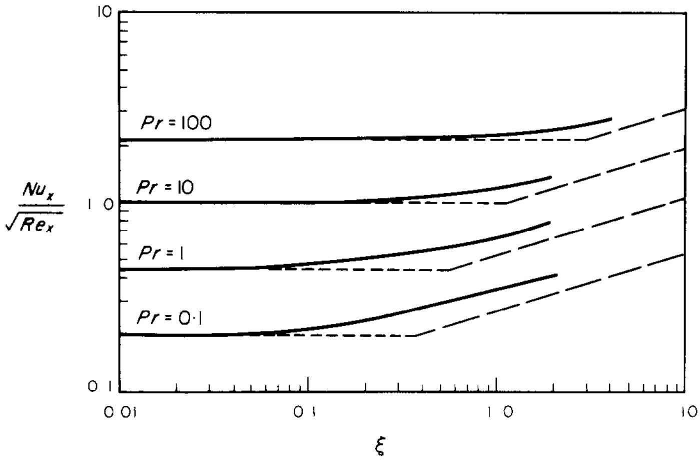


FIG. 1.
———local similarity solution.
· · · · · · · pure forced convection.
– – – – – pure forced convection.

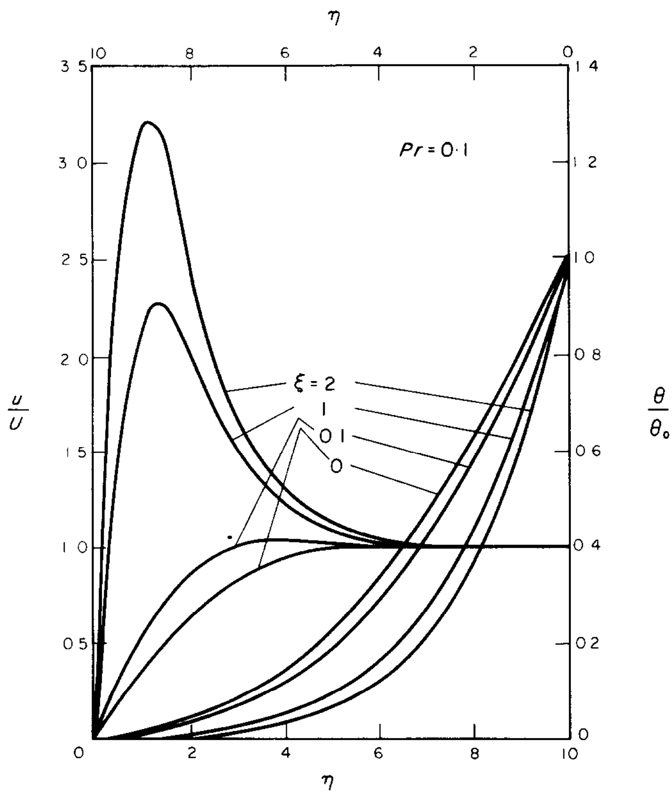


FIG. 2.

Table 1

$\left(\frac{Gr_x^{*2}}{Re_x^5}\right)^{\frac{1}{2}}$	$Pr = 0{\cdot}1$ $\left(\frac{\partial^2 f}{\partial \eta^2}\right)_{\xi,0}$	$-\theta(\xi, 0)$	$Pr = 0{\cdot}72$ $\left(\frac{\partial^2 f}{\partial \eta^2}\right)_{\xi,0}$	$-\theta(\xi, 0)$	$Pr = 1$ $\left(\frac{\partial^2 f}{\partial \eta^2}\right)_{\xi,0}$	$-\theta(\xi, 0)$	$Pr = 10$ $\left(\frac{\partial^2 f}{\partial \eta^2}\right)_{\xi,0}$	$-\theta(\xi, 0)$	$Pr = 100$ $\left(\frac{\partial^2 f}{\partial \eta^2}\right)_{\xi,0}$	$-\theta(\xi, 0)$
0	0·33257	4·93984	0·3326	2·4375	0·3326	2·1775	0·3326	1·0016	0·3326	0·4644
0·1	0·58563	4·47398	0·4209	2·3258	0·4060	2·0916	0·3510	0·9903	0·3367	0·4632
1	3·53855	2·88811	1·7173	1·6375	1·5341	1·5015	0·7457	0·8270	0·4497	0·4340
2	6·56319	2·40940	3·0908	1·3791	2·7414	1·2678	1·2253	0·7216	0·6175	0·4014
4									0·9721	0·3552

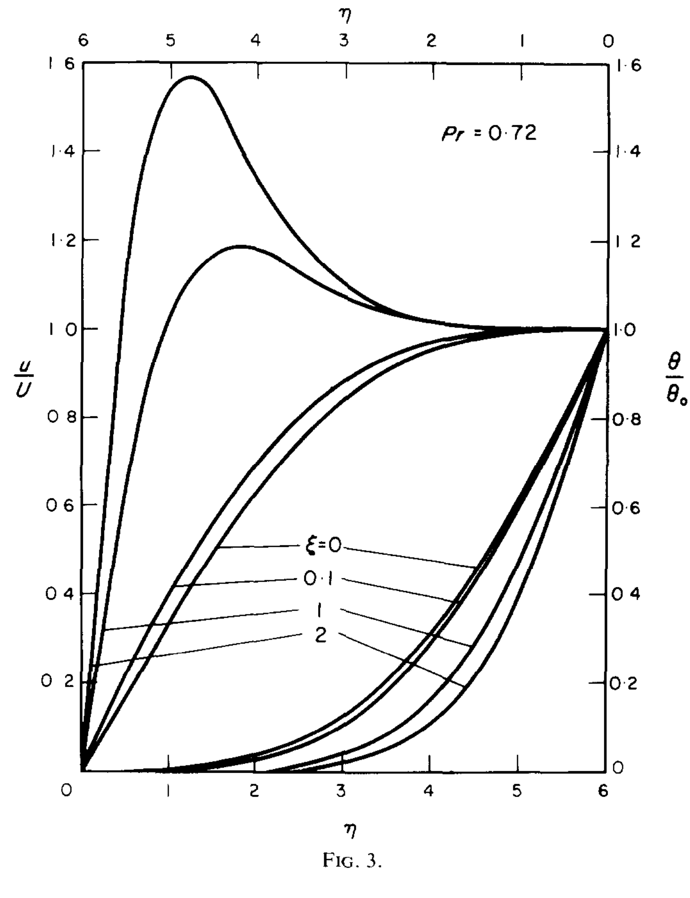


FIG. 3.

2. THE EQUATIONS

Under the usual assumptions the equations expressing conservation of mass, momentum and energy take the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(T - T_0) + \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2}. \tag{3}$$

As posed the boundary conditions of the problem are

$$u = v = 0, \frac{\partial T}{\partial y} = \frac{-q}{k} \quad \text{on } y = 0$$
$$u \to U, \quad T \to T_0 \quad \text{as } y \to \infty \tag{4}$$
$$u = U, \quad T = T_0 \quad \text{at } x = 0, y > 0.$$

The transformations consistent with the forced convection nature of the flow in the vicinity of the leading edge are

$$\psi = (\nu U x)^{\frac{1}{2}} f(\xi, \eta)$$
$$T - T_0 = \frac{-q}{k}\left(\frac{\nu x}{U}\right)^{\frac{1}{2}} \theta(\xi, \eta) \tag{5}$$
$$\eta = y\sqrt{\left(\frac{U}{\nu x}\right)}; \quad \xi = \left(\frac{g^2 \beta^2 q^2 \nu}{k^2 U^5}\right)^{\frac{1}{2}} x.$$

The implementation of these transformations and the application of the local similarity principle yield,

$$f''' + \tfrac{1}{2} f f'' - \xi^{\frac{1}{2}} \theta = 0 \tag{6}$$

$$\frac{1}{Pr}\theta'' + \tfrac{1}{2}\{f\theta' - \theta f'\} = 0 \tag{7}$$

where dashes are associated with η derivatives, to be solved subject to the boundary conditions

$$f(\xi, 0) = \left(\frac{\partial f}{\partial \eta}\right)_{\xi, 0} = 0, \left(\frac{\partial \theta}{\partial \eta}\right)_{\xi, 0} = 1$$
$$\left(\frac{\partial f}{\partial \eta}\right)_{\xi, \infty} = 1, \theta(\xi, \infty) = 0. \tag{8}$$

At any streamwise position along the plate the value of ξ may be regarded as an assignable constant parameter and equations (6) and (7) as a system of ordinary differential equations at each streamwise location of interest.

3. RESULTS

Equations (6) and (7) have been solved numerically for the range of Prandtl numbers $Pr = 0{\cdot}1$, 0·72, 1, 10, 100 and for ranges of ξ encompassing the transition from a purely forced

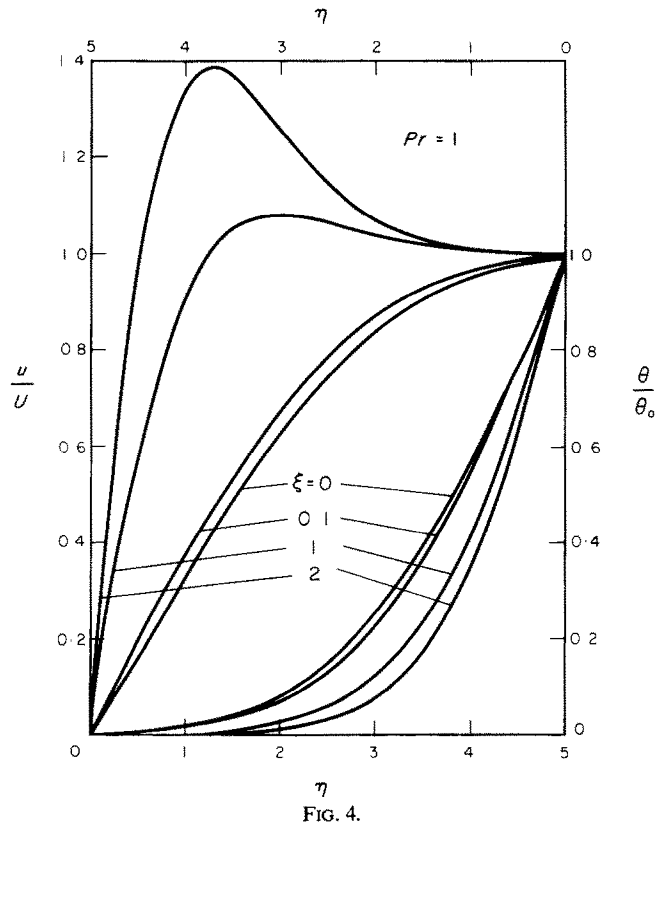
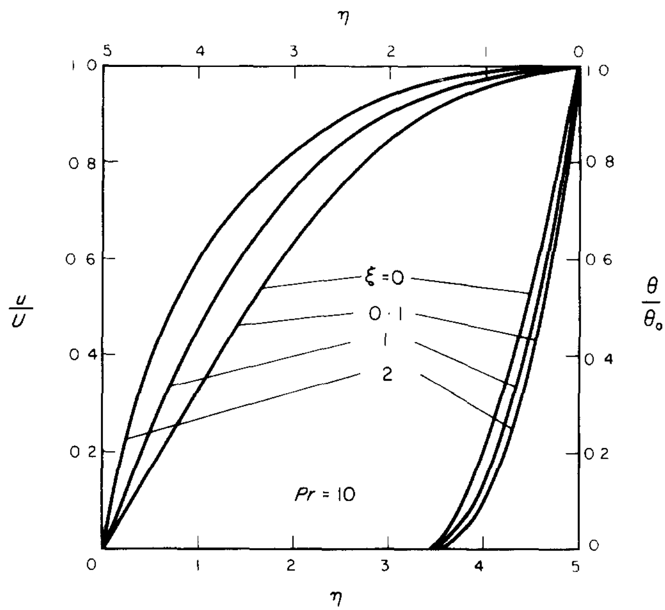
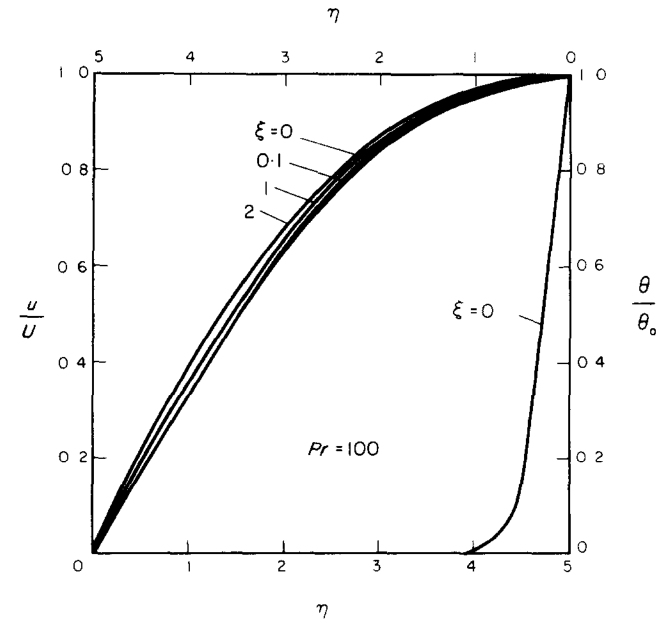


FIG. 4.



η
5
4
3
2
1
0
1·0
0·8
0·6
0·4
0·2
0
u/U
θ/θ₀
ξ = 0
0·1
1
2
Pr = 10
η

FIG. 5.



η
5
4
3
2
1
0
1·0
0·8
0·6
0·4
0·2
0
u/U
θ/θ₀
ξ = 0
0·1
1
2
ξ = 0
Pr = 100
η

FIG. 6.

convection region to one in which buoyancy forces are expected to predominate. Successful integration of (6) and (7) is dependent on a knowledge of the two unknown boundary conditions at $\eta = 0$, namely $(\partial^2 f/\partial \eta^2)_{\xi, 0}$ and $(\theta)_{\xi, 0}$, which must be determined such that integration of (6) and (7) from $\eta = 0$ leads to velocity and temperature profiles fulfilling the boundary conditions away from the plate. These numerical values for various values of Pr and ξ are presented in Table 1.

In terms of the variables introduced the local Nusselt number may be expressed as $Nu_x/\sqrt{(Re_x)} = -1/\theta(\xi, 0)$. This quantity has been plotted in Fig. 1 where for each Prandtl number it is set against the associated pure forced convection asymptote and the associated pure free convection asymptote, obtained from Sparrow and Gregg [5]. The forecast of local Nusselt number obtained from a local similarity hypothesis was demonstrated, by Lloyd and Sparrow [1], to provide a useful estimate in the region of transition between asymptotes for the case $Pr = 0{\cdot}72$ where corroborative evidence was available in the form of experimental results and series solutions. It was then reasonably inferred that their results might provide useful estimates of the local heat transfer in the transition region for other values of Prandtl number. In the constant heat flux case, although series solutions or experimental results are not available, it seems reasonable to suppose that the application of local similarity is equally appropriate and moreover that plots of local Nusselt number in Fig. 1 will provide useful estimates for the given physical situations.

Graphical representations of velocity and temperature profiles are presented simultaneously for $Pr = 0{\cdot}1$, $0{\cdot}72$, 1, 10, 100 in Figs. 2–6 respectively. Velocity profiles are referred to the left and lower axes and temperature profiles are referred to right and upper axes. As the Prandtl number decreases buoyancy effects are increasingly in evidence in velocity profiles and strong overshoot is a feature at the lower end of the range. Variations in temperature profiles are far less pronounced over the range of Prandtl numbers and indeed at $Pr = 100$ profiles at various ξ are so close to each other that Fig. 6 contains only the temperature profile at $\xi = 0$ as typical for that value of Prandtl number.

4. CONCLUDING REMARKS

Local Nusselt number forecasts towards the free convection limit are still lacking. It may well be that a straightforward extrapolation of the local similarity forecast would be a good estimate. With the problem formulated in terms of ξ it should be possible to investigate this suggestion via a coordinate expansion solution downstream. The problem is not trivial however since the question arises as to the contributions of eigen-solutions associated with the presence of the leading edge.

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